KEY

Derivatives of Trigonometric Functions

- **Trig. Function Rules**
  - If $y = f(x) = \sin x$, then $y' = \frac{dy}{dx} = f'(x) = \cos x$
  - If $y = f(x) = \cos x$, then $y' = \frac{dy}{dx} = f'(x) = -\sin x$

Proof of Sine Derivative

Limit Definition for sin:

$$\frac{d}{dx}\sin(x) = \lim_{h\to 0}\frac{\sin(x+h)-\sin(x)}{h}$$

Using angle sum identity, we get

$$\frac{d}{dx}\sin(x) = \lim_{h\to 0}\frac{[\sin(x)\cos(h)+\cos(x)\sin(h)]-\sin(x)}{h}$$

Rearrange the limit so that the sin(x)'s are next to each other

$$\frac{d}{dx}\sin(x) = \lim_{h\to 0}\frac{\cos(x)\sin(h)-\sin(x)+\sin(x)\cos(h)}{h}$$

Factor out a sin from the quantity on the right

$$\frac{d}{dx}\sin(x) = \lim_{h\to 0}\frac{\cos(x)\sin(h)-\sin(x)(1-\cos(h))}{h}$$

Separate the two quantities and put the functions with x in front of the limit (We are only concerned with the limit of h)

$$\frac{d}{dx}\sin(x) = \cos(\mathrm{x})\left(\lim_{h\to 0}\frac{\sin(h)}{h}\right) - sin(x)\left(\lim_{h\to 0}\frac{1-\cos(h)}{h}\right)$$

We can plug in 1 and 0 for the limits and get cos(x)

$$\frac{d}{dx}\sin(x) = \cos(x)(1) - sin(x)(0)$$

$$\frac{d}{dx}\sin(x) = \cos(x)$$

The proof of the cosine derivative is similar.

- **Examples**

1. Find y' for $y = x + \frac{\cos x}{2}$

$y = x + \frac{1}{2}\cos x$

$y' = 1 + \frac{1}{2}(-\sin x)$

$\boxed{y' = 1 - \frac{1}{2}\sin x}$

2. Find $\frac{dy}{dx}$ for $y = 2x\cos x + 2\sin^2 x$

$y = 2x\cos x + 2(\sin x)^2$

$y' = 2x(\cos x)' + \cos x(2x)' + 4(\sin x)(\sin x)'$

$y' = 2x(-\sin x) + \cos x(2) + 4(\sin x)(\cos x)$

$\boxed{y' = -2x\sin x + 2\cos x + 4\sin x\cos x}$

3. Find $g'(x)$ for $g(x) = \sqrt{\cos x \sin^2 x}$

$g(x) = (\cos x \sin^2 x)^{1/2}$

$g(x) = (\cos x)^{1/2}(\sin^2 x)^{1/2}$

$g(x) = (\cos x)^{1/2}(\sin x)$

$g'(x) = ((\cos x)^{1/2})(\sin x)' + (\sin x)((\cos x)^{1/2})'$

$g'(x) = ((\cos x)^{1/2})(\cos x) + (\sin x)(\frac{1}{2}(\cos x)^{-1/2})(-\sin x)$

4. Find $\frac{dy}{dx}$ given $y = \log_3\left(\frac{3e^{-x}}{\sin^2 x}\right)$

$y = \log_3(3e^{-x}) - \log_3(\sin x)^2$

$y = (\log_3 3) + (\log_3 e^{-x}) - 2\log_3(\sin x)$

$y = 1 - x\log_3 e - 2\log_3 \sin x$

$\rightarrow y' = 0 - \log_3 e - 2\left(\frac{1}{\ln 3}\right)\left(\frac{\cos x}{\sin x}\right)$

5. Find $h'(x)$ for $h(x) = \frac{\sin x}{\cos x} \rightarrow \boxed{\tan x}$

$$h'(x) = \frac{\cos x(\sin x)' - (\sin x)(\cos x)'}{(\cos x)^2} = \frac{\cos x(\cos x) - (\sin x)(-\sin x)}{(\cos x)^2}$$

$$= \frac{\cos^2 x + \sin^2 x}{(\cos x)^2} = \frac{1}{(\cos x)^2} = \boxed{\sec^2 x}$$

More Trig Rules:

1. $\frac{d}{dx}[\tan x] = \sec^2 x$
2. $\frac{d}{dx}[\cot x] = -\csc^2 x$
3. $\frac{d}{dx}[\sec x] = \sec x \tan x$
4. $\frac{d}{dx}[\csc x] = -\csc x \cot x$

Examples: Find the derivative of the given function.

$\rightarrow$ chain

$(\csc x)^2$

1. $f(x) = \frac{x-\tan x}{\csc^2 x+\sec x}$

$$f'(x) = \frac{(\csc^2 x + \sec x)(x - \tan x)' - (x - \tan x)(\csc^2 x + \sec x)'}{(\csc^2 x + \sec x)}$$

$$\frac{(\csc^2 x + \sec x)(1 - \sec^2 x) - (x - \tan x)(2\csc x(-\csc x \cot) + \sec x \tan x)}{(\csc^2 x + \sec x)}$$

2. $g(x) = e^{x\cot x}$ $\quad e^{f(x)}$

$$g'(x) = (e^{x\cot x})(x\cot x)'$$

$$g'(x) = (e^{x\cot x})((x)(-\csc^2 x) + (\cot x)(1))$$

3. $y = \sqrt[3]{(\sec^2 x)(\cos x)}$ —— or $y = \sqrt[3]{\frac{\cos x}{\cos^2 x}} = \sqrt[3]{\frac{1}{\cos x}} = \sqrt[3]{\sec x}$

$= (\sec x)^{1/3}$

$$\boxed{y' = \frac{1}{3}(\sec x)^{-2/3}(\sec x \tan x)}$$

$$y = ((\sec x)^2 (\cos x))^{1/3}$$

$$y = (\sec x)^{2/3} (\cos x)^{1/3}$$

$$y' = (\sec x)^{2/3} ((\cos x)^{1/3})' + (\cos x)^{1/3}((\sec x)^{2/3})'$$

$$(\sec x)^{2/3}\left(\tfrac{1}{3}(\cos x)^{-2/3}\right)(-\sin x)) + (\cos x)^{1/3}\left(\tfrac{2}{3}(\sec x)^{-1/3}\right)(\sec x \tan x)$$

4. $y = \sin(2x) + \tan\left(\frac{x}{2}\right)$

$$y = 2\sin x\cos x + \frac{1-\cos x}{\sin x}$$

$$y' = 2\sin x(\sin x) + (\cos x)(-2\sin x) + \frac{(\sin x)(\sin x) - (1-\cos x)(\cos x)}{(\sin x)^2}$$