Exponential and Logarithmic Functions

The inverse of $f(x) = b^x$ is called a logarithmic function with base *b*, denoted $\log_b x$, and read log base b of x.

<u>**Relating Logarithmic and Exponential Forms**</u>

$$\log_{\underset{\text{base}}{b}} x = \underset{\text{exponent}}{y} \quad \text{if and only if} \quad \underset{\text{base}}{b}^{\overset{\text{exponent}}{y}} = x$$

The statement above indicates that $\log_b x$ is defined as the exponent to which *b* must be raised in order to produce *x*. Therefore, when evaluating a logarithm, remember that a logarithm is an exponent.

There are many logarithms that you can evaluate without the use of a calculator.

Example #1: Evaluate $\log_3 81$

Step #1: Set the log equal to *x*. $\log_3 81 = x$

Step #2: Exponentiate $3^x = 81$

Step #3: Write both sides with the same base. $3^x = 3^4$

Step #4: Drop the bases and solve for *x.* $x = 4$

Example #2: Evaluate $\log_5 \sqrt{5}$

Example #3: Evaluate $\log_7 \frac{1}{49}$

Example #4: Evaluate $\log_{16} \sqrt{2}$

Special Types of Logarithms:

1. Common log: log with a base of 10 written as $\log x$ but means $\log_{10} x$

2. Natural log: log with a base of *e* written as $\ln x$ but means $\log_e x$

   e is an irrational number approximately equal to 2.71828

These special logs can be evaluated using a calculator. The common log button says LOG and the natural log says LN.

Example #5: Evaluate $\log 0.001$

You can simply type this into your calculator since it is a common log, or to complete the problem by hand. (Note you will not be able to complete every problem by hand).

Rewrite as	$\log_{10} 0.001 = x$
Write as an exponent:	$10^x = 0.001$
Write both sides with the same base:	$10^x = 10^{-3}$
Drop the bases and solve for x:	$x = -3$

Note:

$$\log_b b = 1$$
$$\log_b 1 = 0$$
$$\ln 1 = 0$$
$$\ln e = 1$$

Example #6: Evaluate $\log 26$

Example #7: Evaluate $\log(-5)$

Example #8: Evaluate $\log 10{,}000$

Example #9: Evaluate $\ln 4$

Example #10: Evaluate $\ln 32$

Recall Properties of Exponents:

1. $b^x \cdot b^y = b^{x+y}$

2. $\frac{b^x}{b^y} = b^{x-y}$

3. $(b^x)^y = b^{xy}$

Since logarithms and exponents have an inverse relationship, these properties of exponents imply these corresponding properties of logarithms.

1. Product Property: $\log_b xy = \log_b x + \log_b y$

2. Quotient Property: $\log_b \frac{x}{y} = \log_b x - \log_b y$

3. Power Property: $\log_b x^p = \text{p} \log_b x$

Example #11: Evaluate $\log_4 \sqrt[5]{64}$

Since the base of the log is 4, we want to express $\sqrt[5]{64}$ with a base of 4 to some power.

$$\sqrt[5]{64} = 64^{\frac{1}{5}} = (4^3)^{\frac{1}{5}} = 4^{\frac{3}{5}}$$

So: $\log_4 \sqrt[5]{64} = \log_4 4^{\frac{3}{5}} = \frac{3}{5}\log_4 4 = \frac{3}{5}(1) = \frac{3}{5}$

Example #12: Evaluate $5lne^2 - lne^3$

Example #13: Evaluate $\log_6 \sqrt[3]{36}$

Example #14: Evaluate $2\log_3 \sqrt{27}$

The properties of logarithms provide a way of expressing logarithmic expressions in forms that use simpler operations, converting multiplication into addition, division into subtraction, and powers and roots into multiplication.

Example #15: Expand $\log 12x^5y^{-2}$

Multiplication expands to addition: $\log 12 + \log x^5 + \log y^{-2}$

Exponents can be rewritten as coefficients in front of each log: $\log 12 + 5\log x - 2\log y$

Example #16: Expand $\ln \frac{x^2}{\sqrt{4x+1}}$

Example #17: Expand $\log_{13} 6a^3bc^4$

The same methods used to expand logs can be used to condense them.

Example #18: Condense $4\log_3 x - \frac{1}{3}\log_3(x+6)$

Since these have the same base (3), we can condense them into one log.

Start by writing the coefficients as exponents: $\log_3 x^4 - \log_3(x+6)^{\frac{1}{3}}$

Subtraction condenses to subtraction: $\log_3 \frac{x^4}{(x+6)^{\frac{1}{3}}} = \log_3 \frac{x^4}{\sqrt[3]{x+6}}$ at this point you should rationalize the denominator.

Example #19: Condense $6\ln(x-4) + 3\ln x$

Example #20: Condense $-5\log_2(x+1) + 3\log_2(6x)$

Change of Base Formula: Sometimes you may need to work with a logarithm that has an inconvenient base. For example, evaluating $\log_3 5$ presents a challenge because calculators have no key for evaluating base 3 logarithms. The change of base formula provides a way of changing such an expression into a quotient of logarithms with a different base.

$$\boldsymbol{\log_b x = \frac{\log_a x}{\log_a b}} \quad \textbf{or} \quad \boldsymbol{\frac{ln\ x}{ln\ b}}$$

Example #21: Evaluate: $\log_3 5$

Use the change of base formula: $\frac{log\ 5}{log\ 3}$

Use a calculator to evaluate since they are in base 10: $\frac{log\ 5}{log\ 3} = \mathbf{1.47}$

Example #22: Evaluate: $\log_{\frac{1}{2}} 6$

Example #23: Evaluate: $\log_{15} 33$

<u>One-to-One Property of Exponential Functions</u>

If $3^x = 3^5$, then $x = 5$

If $log\ \ x = 3$, then $10^3 = x$

Example #24: Solve the equation: $36^{x+1} = 6^{x+6}$

Example #25: Solve the equation: $\left(\frac{1}{2}\right)^c = 64^{\frac{1}{2}}$

Example #26: Solve the equation: $16^{x+3} = 4^{4x+7}$

Example #27: Solve the equation: $\left(\frac{2}{3}\right)^{x-5} = \left(\frac{9}{4}\right)^{\frac{3x}{4}}$

Exponentiating

Example #28: $ln\, x = 6$

Example #29: $6 + 2\log 5x = 18$

Example #30: $\log_8 x^3 = 12$

Example #31: $-3\ln x = -24$

Example #32: $4 - 3\log(5x) = 16$

Example #33: $\log_{\mathbf{3}}(x^2 - 1) = 4$

Solving Exponential Equations (when you cannot get a like base on both sides)

Example #34: $4^x = 13$

Example #35: $8^x = 0.165$

Example #36: $1.43^x + 3.1 = 8.48$

Example #37: $e^{2+5x} = 12$

Example #38: $e^{4-3x} = 6$

More logarithmic equations

Example #39: $\ln(x+2) + \ln(3x-2) = 2\ln 2x$

Example #40: $\log_{12} 12x + \log_{12}(x-1) = 2$